Problem 1 Let:

LAST NAME: Dec 9/2013

FIRST NAME: Solution

$$L = \{a^n d^\ell a^p b^k c^j a^m \mid k = 2\ell,\ p = 3j,\ m = n = 0,\ n, k, \ell, j, m, p \geq 0\}$$

(⊛) $L = \{d^\ell a^{3j} b^{2\ell} c^j\}$ template ↗

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer: Impossible, L is not context free. To prove it, assume the opposite. Let k be the constant in the P. Lemma. Select $\ell, j > k$ and word $w_0 = d^\ell a^{3j} b^{2\ell} c^j$. Since the pumping window is shorter than k, it must reside within a single-letter segment or no more than two adjacent ones. If pumped up, w_0 violates template (⊛).

(b) Draw a state transition graph of a finite automaton that accepts L. If such an automaton does not exist, prove it.

Answer: Impossible. L is not regular. If L was regular, it would be context-free, since every regular language is context-free. Since the answer to part (a) implies that L is not context-free, it cannot be regular.

Problem 2 Let:

LAST NAME:

FIRST NAME: Solution

$$L = \{c^n b^k a^p d^\ell a^j d^m \mid j = 2\ell+1,\ m = 3n+2,\ k = p = 0,\ n, k, \ell, j, m, p \geq 0\}$$

$L = \{c^n d^\ell a^{2\ell+1} d^{3n+2}\}$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

Answer:

$G = (V, \Sigma, P, S)$

$V = \{S, A\}$

$\Sigma = \{a, c, d\}$

$P: S \to cSddd \mid Add$

$A \to dAaa \mid a$

(b) Draw a state transition graph of a finite automaton that accepts L. If such an automaton does not exist, prove it.

Answer: Impossible, since L is not regular. Assume the opposite, that L is regular. Observe that in every word of L number of c's say n is related to number of d's to the right of a's, which is then $3n+2$. Let k be the constant in the pumping lemma. Select $n > k$ and a word $w_0 = c^n a d^{3n+2}$. In a pumping decomposition, pumping window is shorter than $k < n$, so entirely within c's. Pump up once $\Rightarrow$ too few d's.

Problem 3 Let:

LAST NAME:

FIRST NAME: Solution

$$L = \{d^n b^k d^j c^\ell a^p c^m \mid m = 2\ell+1,\ j = 3n+2,\ p = k = 0,\ n, k, \ell, j, m, p \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such a grammar does not exist, prove it.

template:
$L = \{d^{4n+2} c^{3\ell+1} \mid n, \ell \geq 0\}$

Answer:

$G = (V, \Sigma, P, S)$

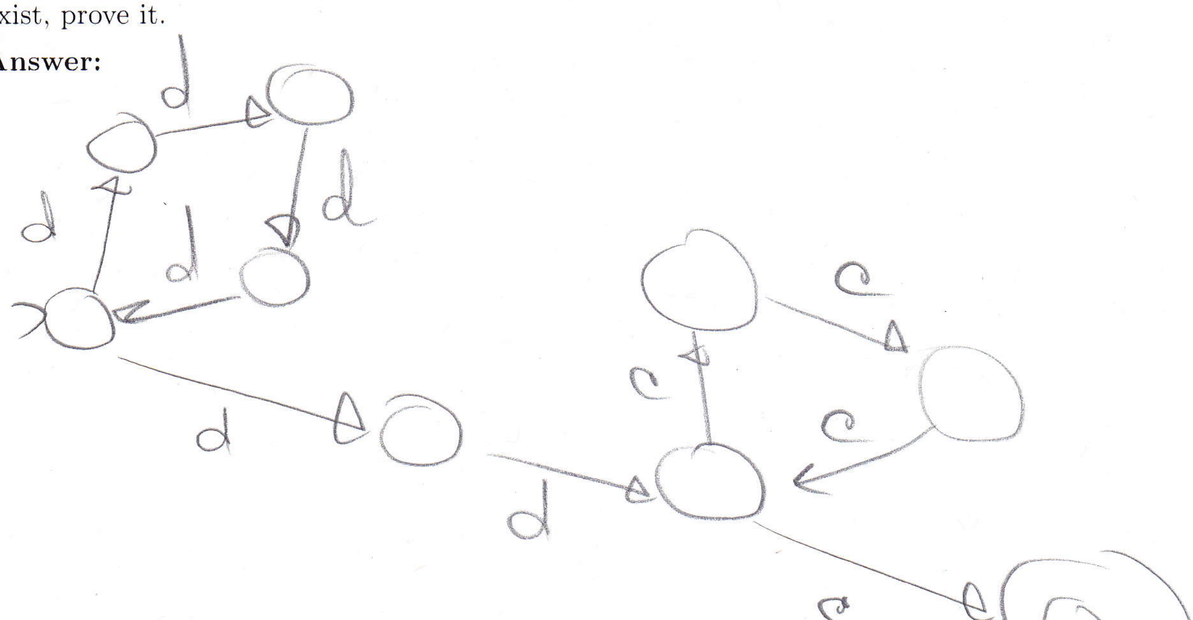
$V = \{S, A, B\}$

$\Sigma = \{c, d\}$

$P: S \to AB$
$A \to ddddA \mid dd$
$B \to cccB \mid c$

(b) Draw a state transition graph of a finite automaton that accepts L. If such an automaton does not exist, prove it.

Answer:



LAST NAME: ______________________

FIRST NAME: Solution

Problem 4 Consider the Turing machine $M = (Q, \Sigma, \Gamma, \delta, q)$ such that: $\Sigma = \{0, 1\}$; $\Gamma = \{B, 0, 1\}$; $Q = \{q, r, s, p, v, t, z, y\}$; and δ is defined by the following transition set:

$[q, 1, r, 1, R]$	$[v, 1, z, 0, L]$
$[r, 1, s, 1, R]$	$[v, 0, y, 0, R]$
$[s, 1, t, 1, R]$	
	$[z, 0, y, 0, R]$
$[t, 0, p, 0, R]$	$[z, 1, y, 1, L]$
$[t, 1, p, 1, R]$	
	$[y, 0, y, 0, R]$
$[p, 0, p, 0, R]$	$[y, B, y, B, R]$
$[p, 1, p, 1, R]$	
$[p, B, v, B, L]$	

(Assume that M is defined so as to have an one-way infinite tape (infinite to the right only.) B is the designated blank symbol.)

Let L be the set of strings on which M *diverges.*

(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer: See (b)

(b) Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$111(0 \cup 1)^*(0 \cup 01)$

(c) Dangerous Professor has told her students to write a program that operates as follows:

INPUT: String w over $\{0, 1\}$.

OUTPUT: **yes** if w is a string that represents a Turing Machine which accepts exactly those strings that belong to the set L (defined at the beginning of this problem);
no otherwise.

Explain the algorithm that should be employed by this program, or state that it does not exist, and prove it.

Answer:

Impossible. If this algorithm existed it would decide the set of machines whose languages have the property "is equal to L". This property is nontrivial since L has it, but $\overline{L}$ does not have it. Hence, by Rice's theorem, the construction is impossible.

LAST NAME:

FIRST NAME: Solution

Problem 5 Consider the Turing machine $M = (Q, \Sigma, \Gamma, \delta, q)$ where: $\Sigma = \{0, 1\}$; $\Gamma = \{1, 0, B\}$; $Q = \{q, p, s, t, v, x\}$; and δ is defined by the following transition set:

$$\begin{array}{ll}
[q,0,q,0,R] & [t,0,v,0,L] \\
[q,1,p,1,R] & [t,1,s,1,R] \\
[q,B,q,B,R] & \\
 & [v,0,x,0,L] \\
[p,0,p,0,R] & [v,1,s,1,R] \\
[p,1,t,1,L] & \\
[p,B,p,B,R] & [s,0,s,0,R] \\
 & [s,1,s,1,R] \\
 & [s,B,s,B,R]
\end{array}$$

(Assume that M is defined so as to have an one-way infinite tape (infinite to the right only.) B is the designated blank symbol.)

Let L be the set of string on which M halts.

(a) List 6 distinct strings that belong to L. If this is impossible, state it and explain why.

Answer: See answer to (b).

(b) Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$0^* 1 0^* 001 (0 \cup 1)^*$

(c) Dangerous Professor has told her students to write a program that operates as follows:

INPUT: String w over $\{0, 1\}$.

OUTPUT: **yes** if w is an element of the set of exactly those strings on which the Turing Machine M (defined at the beginning of this problem) halts; **no** otherwise.

Explain the algorithm that should be employed by this program, or state that it does not exist and prove it.

Answer:

Convert the regular expression constructed in part (b) to a finite automaton, and then this automaton into a deterministic finite automaton. Simulate the latter automaton and decide exactly as it does.

LAST NAME:

FIRST NAME: Solution

Problem 6 Consider the Turing machine $M = (Q, \Sigma, \Gamma, \delta, q, F)$ where: $\Sigma = \{0, 1\}$; $\Gamma = \{B, 0, 1\}$; $Q = \{q, s, x, p, y, z, v, m\}$; $F = \{v\}$; and δ is defined by the following transition set:

$[q, 0, q, 0, R]$	$[y, 1, z, 1, L]$
$[q, 1, s, 1, R]$	$[y, 0, x, 0, R]$
$[q, B, x, B, R]$	
	$[z, 1, v, 1, R]$
$[s, 0, q, 0, R]$	$[z, 0, m, 0, R]$
$[s, 1, p, 1, R]$	
$[s, B, x, B, R]$	$[x, 0, x, 0, R]$
	$[x, 1, x, 1, R]$
$[p, 0, p, 0, R]$	$[x, B, x, B, R]$
$[p, 1, p, 1, R]$	
$[p, B, y, B, L]$	

(Assume that M is defined so as to have an one-way infinite tape (infinite to the right only.) M accepts by final state. B is the designated blank symbol.)

Let L_1 be the set of string which M accepts.
Let L_2 be the set of string which M rejects.

(a) Write a regular expression that defines L_1. If such a regular expression does not exist, prove it.

Answer:

$(0 \cup 1)^* 11 (0 \cup 1)^* 11$
$\cup$
$11 \cup 111$

(b) Write a regular expression that defines L_2. If such a regular expression does not exist, prove it.

Answer:

$(0 \cup 1)^* 11 (0 \cup 1)^* 01$

(c) Dangerous Professor has told her students to write a program that operates as follows:

INPUT: String w over $\{0, 1\}$.

OUTPUT: **yes** if w is a string such that the Turing Machine represented by w halts exactly when the machine M (defined at the beginning of this problem) rejects;
no otherwise.

Explain the algorithm that should be employed by this program, or state that it does not exist and prove it.

Answer: Impossible. If this algorithm existed, it would decide the set of TMs whose languages have the property "is equal to L_2". Since L_2 has the property but L_1 does not the property is nontrivial and the construction is impossible by Rice's theorem.

Note: If M is defined so that $F = \{m\}$, swap answers to (a) and (b).

LAST NAME:

FIRST NAME: Solution

Problem 7 Let D_{bc} be the set of all palindromes which consist only of letters $\{b, c\}$ and have an even length which is greater than or equal to 2.

Let L be the set of all strings over the alphabet $\{a, b, c\}$ which satisfy all of the following properties.

1. the string never contains more than one a;
2. if the string does not contain a, then it is an element of D_{bc};
3. if the string contains a, it would be an element of D_{bc} if the a was removed.

Write a complete formal definition of a context-free grammar that generates the language L. If such a grammar does not exist, state it and explain why.

Answer:

$G = (V, \Sigma, P, S)$

$\Sigma = \{a, b, c\}$

$V = \{S, D, A\}$

$P:$ $S \to D \mid A$

$D \to bDb \mid cDc \mid bb \mid cc$

$A \to bAb \mid cAc \mid$
$abDb \mid baDb \mid bDab \mid bDba \mid$
$acDc \mid caDc \mid cDac \mid cDca$